

# On the interactions of Skyrmions with domain walls

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## Abstract

We study classical solutions of a particular version of the modified Skyrme model in  $(3+1)$  dimensions. The model possesses Skyrmion solutions as well as stable domain walls that connect different vacua of the theory. We show that there is an attractive interaction between Skyrmions and domain walls. Thus Skyrmions can be captured by the domain walls. We show also that, when the mass term is of a special type, the model possesses bound states of Skyrmions and of the domain wall. They look like deformed 2-dimensional Skyrmions captured by the wall. The field configurations of these solutions can be interpreted as having come from the evolution of the 3-dimensional Skyrmions captured by the domain wall. For more conventional choices of the mass term of the model in the model the attraction between the Skyrmions and the wall leads to the capture of the Skyrmions which are then turned into topological waves which spread out on the wall. We have observed, numerically, such captures and the emission of the waves. We speculate that this observation may be useful in the explanation of the problem of baryogenesis and baryon-antibaryon asymmetry of the Universe.

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## I. INTRODUCTION

The Skyrme model in (3+1) dimensions, originally introduced by Skyrme [1], has attracted a lot of attention as a reasonable good effective theory for describing baryons [2]. Baryons in this theory are described by topologically nontrivial classical configurations (Skyrmions). Recently there has been some discussion of the problem of the baryon-antibaryon asymmetry of the Universe. This problem may be related to the problem of interaction between Skyrmions (or baryons) and domain walls during the early stages of the Universe.

The interaction of particles with domain walls has become a subject of interest following a seminal paper by Voloshin [3]. A detailed analysis of the scattering of Abelian gauge particles on domain walls may be found in a paper by Farrar et al [4]. The problem of the chiral fermion determinant in the presence of a domain wall was discussed in [5]. It is also worth mentioning that the expanding bubbles of a new phase during the electroweak phase transition may be an additional source for the CP - violation effects of the electroweak baryogenesis, see, *e.g.* [6,7].

Recent interest in the problem of the interaction of particles with domain walls was generated by a paper of Dvali et. al [8]. In this paper the authors consider the interaction of monopoles with domain walls. They suggest that this interaction may help to solve the problem of low density of monopoles in the Universe. They conjecture that the domain walls would sweep up all the monopoles which would then defuse along the domain walls.

Very recently this conjectured interaction of monopoles with domain walls was supported by numerical stimulations of monopole-domain wall collision processes as reported in [9].

The fact that extended objects, such as Skyrmions, interact nontrivially with domain walls had already been discussed in our previous paper [10]. In that paper we studied the interaction of Skyrmions with domain walls in a two-dimensional baby-Skyrmion model [11]. In [10] we showed that domain walls can attract and absorb Skyrmions. After its collision with a domain wall the Skyrmion splits into two parts which propagate (with the speed of light) along the domain wall. Each part of the former Skyrmion looks like a wave packet which carries one half of the topological charge of the Skyrmion. That is why the compound objects like Skyrmions and monopoles can interact nontrivially with domain walls. This

interaction requires further careful study.

In this paper we concentrate our attention on the problem of the interaction of Skyrmions with domain walls in a realistic 3-dimensional case. To do this we consider the conventional Skyrme model [1,2] but with a slightly modified expression for the mass term. Our modification is motivated by the existence of solutions in the form of domain walls. In fact, first we introduce the simplest modification which is then further altered by the introduction of an additional parameter. The model is introduced and discussed in detail in the next section. For our choice of the mass term it possesses solutions which correspond to both Skyrmions and domain walls. Next we study the problem of their interaction. We demonstrate that there exists a solution, which may be considered as a bound state of a Skyrmion and a domain wall. We also show that the Skyrmions and the domain walls attract each other. We have performed a series of numerical simulations studying this interaction. In our simulations we have always found the attraction although the final stage of the evolution process(*ie* whether the topological waves on the wall spread out or oscillate) depends on the details of the modified mass term of the model. In each case the Skyrmions are captured, resulting in the emission of topological waves. The waves spread out on the wall in the model with the simplest version of the modified mass term, or oscillate when the mass term is further modified to prevent this spreading.

## II. THE MODEL

To study the interaction between Skyrmions and domain walls in (3+1) dimensions we consider a model described by the Lagrangian density

$$L = L_{kin} + L_{Sk} + L_{mass}^{mod}, \quad (1)$$

where  $L_{kin}$  and  $L_{Sk}$  are the standard kinetic and Skyrme terms for the effective chiral field theory [1,2] and so are given by:

$$L_{kin} = \frac{F_\pi^2}{16} \text{Tr}(UU^+), \quad (2)$$

$$L_{Sk} = \frac{1}{32e^2} \text{Tr}[\partial_\mu UU^+, \partial_\nu UU^+]^2. \quad (3)$$

Here  $U$  is an  $SU(2)$   $2 \times 2$  matrix,  $U = \Phi_0 + i\vec{\tau}\vec{\Phi}$  and  $\Phi_0$  and  $\vec{\Phi}$  are real scalar fields, which satisfy the constraint

$$\Phi_0^2 + \vec{\Phi}^2 = 1. \quad (4)$$

The third term, the mass term, violates chiral symmetry and is very nonunique, as it is determined only in the limit of small pion fields. In our case we use the following form of the mass term:

$$L_{mass}^{mod} = \frac{1}{64} m_\pi^2 F_\pi^2 Tr[(U + U^\dagger - 2)(U + U^\dagger + 2)]. \quad (5)$$

Our choice (5) of the mass term gives us two different absolute minima for the energy of the field configuration at constant  $U = \pm 1$  (vacua states of the theory). Note that the more usual mass term leads to the existence of only one vacuum state of the theory (at  $U = 1$ ):

$$L_{mass} = \frac{1}{8} m_\pi^2 F_\pi^2 Tr(U + U^\dagger - 2). \quad (6)$$

Note also that when the field configuration differs slightly from the vacuum  $U = 1$ , i.e.  $|U - 1| \ll 1$ , the two mass terms (5) and (6) are very similar.

Both the modified (5) and the standard (6) versions of the Lagrangian possess solutions of the Skyrmion type. However, the modified version of the model possesses also solutions in the form of domain walls. The explicit expression for this solution can be easily obtained after a further change of variables from the  $(\vec{\Phi}, \Phi_0)$  field variables. Namely, we replace the  $(\vec{\Phi}, \Phi_0)$  by  $(f, g, h)$  defined by

$$(\vec{\Phi}, \Phi_0) = (\sin f \sin g \sin h, \sin f \sin g \cos h, \sin f \cos g, \cos f), \quad (7)$$

where  $f = f(\vec{r}, t)$ ,  $g = g(\vec{r}, t)$  and  $h = h(\vec{r}, t)$ ,  $f \in [0, \pi]$ ,  $g \in [0, \pi]$ ,  $h \in [0, 2\pi]$ . Note that the constraint (4) is automatically satisfied. The Lagrangian (1), when written in terms of  $f, g$  and  $h$  is given by:

$$\begin{aligned} \mathcal{L} = & \frac{F_\pi^2}{8} \{ f_\mu^2 + \sin^2 f g_\mu^2 + \sin^2 f \sin^2 g h_\mu^2 + k \sin^2 f [(f_\mu g_\mu)^2 - f_\mu^2 g_\mu^2] \\ & + \sin^2 g ((f_\mu h_\mu)^2 - f_\mu^2 h_\mu^2) + \sin^2 f \sin^2 g ((h_\mu g_\mu)^2 - g_\mu^2 h_\mu^2) \} - 2 m_\mu^2 V(f, g), \end{aligned} \quad (8)$$

where  $k = 2/F_\pi^2 e^2$ .

Consider now the case when fields  $g$  and  $h$  are constant. In this case we get from (5) the following equation of motion for the field  $f$ :

$$f_\mu{}^\mu + \frac{m_\pi^2}{2} \sin 2f = 0. \quad (9)$$

Among solutions of this equation (9) there is the static soliton solution

$$f_{sol}(x) = 2 \arctan \exp(\pm m_\pi(x - x_0)). \quad (10)$$

In the (3+1) dimensional space-time this solution looks like a domain wall which links the two vacuum states  $U = \pm 1$  of the theory.

In terms of the angular variable  $f$  the Skyrmon solution becomes:

$$U(\vec{r}) = \cos f_{Sk}(r) + i\vec{\tau}\vec{n} \sin f_{Sk}(r), \quad (11)$$

where  $f_{Sk}$  is a Skyrmon profile function [2] satisfying the boundary conditions  $f_{Sk}(0) = \pi$ ,  $f_{Sk}(\infty) = 0$  and  $\vec{n} = \vec{r}/r$ . The Skyrmon solutions are characterized by an integer valued degree of the mapping of  $S^3$  (compactified  $R^3$  physical space) into  $S^3$  - isospace. The analytical formula for this degree of mapping (*ie* the winding number), when written in terms of the fields  $f$ ,  $g$  and  $h$ , is

$$Q = -6 \int \sin^2 f \sin g \epsilon_{ijk} f_i g_j h_k dx^3, \quad (12)$$

where we have used Latin indices to denote spatial variables.

### III. A TOPOLOGICAL SOLITON ON THE DOMAIN WALL

As it was shown in our previous paper [10], a (2+1) - dimensional Skyrme-like field theory possesses solutions which correspond to topological waves. They behave like massless wave packets which carry the topological charge. It would be interesting to see whether analogous solutions exist in a (3+1)-field theory.

To answer this question let us seek solutions of the equation of motion for the Lagrangian (5) in the form of a generalized cylindrically symmetric hedgehog ansatz:

$$f = f(t, x, \rho), \quad g = g(t, x, \rho), \quad h = n\varphi, \quad (13)$$

where  $n$  is integer and we have changed Euclidean coordinates  $(x, y, z)$  to the cylindrical ones  $(x, r, \varphi)$ ,

$$(x, y, z) \rightarrow (x, \rho \cos \varphi, \rho \sin \varphi). \quad (14)$$

It is interesting to note that  $h = n\varphi$  is compatible with the equations of motion (ie the  $h$  equation is automatically satisfied). So we have to look at the remaining equations. In terms of functions  $f$  and  $g$  the Lagrangian becomes:

$$\mathcal{L} = \frac{F_\pi^2}{8} \int \{ f_t^2 - f_x^2 - f_\rho^2 + \sin^2 f (g_t^2 - g_x^2 - g_\rho^2) - \sin^2 f \sin^2 g \frac{n^2}{\rho^2} + k \sin^2 f [(f_t g_x - f_x g_t)^2 + (f_t g_\rho - f_\rho g_t)^2 - (f_x g_\rho - f_\rho g_x)^2 + \sin^2 g \frac{n^2}{\rho^2} (f_t^2 - f_x^2 - f_\rho^2)] \} 2\pi \rho d\rho dx, \quad (15)$$

$$+ \sin^2 f \sin^2 g \frac{n^2}{\rho^2} (g_t^2 - g_x^2 - g_\rho^2) - 2m_\mu^2 V(f, g) \} 2\pi \rho d\rho dx, \quad (16)$$

In (16) we have used an expression more general than (5) for the mass term. Our new mass term depends on both variables  $f$  and  $g$ . The purpose of this generalization will become obvious soon. The equations of motion for functions  $f$  and  $g$ , which follow from the Lagrangian (16), are:

$$\begin{aligned} & f_{tt} - f_{xx} - f_{\rho\rho} - \frac{f_\rho}{\rho} - \sin f \cos f (g_t^2 - g_x^2 - g_\rho^2 - \sin^2 g \frac{n^2}{\rho^2}) + m_\mu^2 \frac{\partial V(f)}{\partial f} \\ & + k \sin f \cos f [(f_t g_x - f_x g_t)^2 + (f_t g_\rho - f_\rho g_t)^2 - (f_x g_\rho - f_\rho g_x)^2 + \sin^2 g (f_t^2 - f_x^2 - f_\rho^2) \frac{n^2}{\rho^2} \\ & - 2 \sin^2 f \sin^2 g (g_t^2 - g_x^2 - g_\rho^2) \frac{n^2}{\rho^2}] + k \sin^2 f [(f_{tt}(g_x^2 + g_\rho^2) - (f_{\rho\rho} + \frac{f_\rho}{\rho})(g_x^2 - g_t^2) \\ & - f_{xx}(g_\rho^2 - g_t^2) - 2f_{xt}g_xg_t - 2f_{\rho t}g_\rho g_t + 2f_{x\rho}g_xg_\rho + g_{t\rho}(f_tg_\rho + f_\rho g_t) + g_{tx}(f_tg_x + f_xg_t) \\ & - g_{x\rho}(g_xf_\rho + g_\rho f_x) - g_{tt}(f_\rho g_\rho + f_xg_x) - g_{xx}(f_tg_t - f_\rho g_\rho) - g_{\rho\rho}(f_tg_t - f_xg_x) + \frac{g_\rho}{\rho}(f_xg_x - f_tg_t) \\ & + 2 \sin g \cos g \frac{n^2}{\rho^2} (f_tg_t - f_xg_x - f_\rho g_\rho) + \sin^2 g (f_{tt} - f_{xx} - f_{\rho\rho} + \frac{f_\rho}{\rho}) \frac{n^2}{\rho^2}] = 0. \end{aligned}$$

and

$$\begin{aligned} & g_{tt} - g_{xx} - g_{\rho\rho} - \frac{g_\rho}{\rho} + 2 \cot f (f_tg_t - f_xg_x - f_\rho g_\rho) + \sin g \cos g \frac{n^2}{\rho^2} + \frac{m_\mu^2}{\sin^2 f} \frac{\partial V(f, g)}{\partial g} \\ & + k [f_{t\rho}(g_tf_\rho + g_\rho f_t) + f_{tx}(g_tf_x + g_x f_t) - f_{x\rho}(f_xg_\rho + f_\rho g_x) - f_{tt}(g_\rho f_\rho + g_x f_x) \\ & - f_{xx}(g_tf_t - g_\rho f_\rho) - f_{\rho\rho}(g_tf_t - g_x f_x) + \frac{f_\rho}{\rho}(g_x f_x - g_tf_t) - (g_{\rho\rho} + \frac{g_\rho}{\rho})(f_x^2 - f_t^2) - g_{xx}(f_\rho^2 - f_t^2) \\ & + g_{tt}(f_x^2 + f_\rho^2) - 2g_{xt}f_x f_t - 2g_{\rho t}f_\rho f_t + 2g_{x\rho}f_x f_\rho + \sin^2 f \sin^2 g (g_{tt} - g_{xx} - g_{\rho\rho} + \frac{g_\rho}{\rho}) \frac{n^2}{\rho^2} \\ & + 4 \sin f \cos f \sin^2 g \frac{n^2}{\rho^2} (f_tg_t - f_xg_x - f_\rho g_\rho) + \sin^2 f \sin g \cos g \frac{n^2}{\rho^2} (g_t^2 - g_x^2 - g_\rho^2) \\ & - \sin g \cos g \frac{n^2}{\rho^2} (f_t^2 - f_x^2 - f_\rho^2)] = 0 \end{aligned} \quad (17)$$

It is worth emphasising here, that setting  $h = n\varphi$  has lead to a serious simplification of the problem. In fact, our choice of  $h$  corresponds to using one integral of motion  $n$ , whose conservation reflects the internal symmetry of the solutions of the theory (1)/. Note that by introducing our hegehog ansatz we have been able to reduce the problem to that of having to solve only two coupled equations for functions  $f$  and  $g$  in (2+1) dimensions instead of having to consider the coupled partial differential equations for three functions  $f$ ,  $g$  and  $h$  in (3+1) dimensions.

Note also that in our ansatz the topological charge becomes:

$$Q = -12\pi n \int \sin^2 f \sin g (f_\rho g_x - f_x g_\rho) dx d\rho. \quad (18)$$

We note that this expression is nonzero only when  $n \neq 0$ . Thus to check whether the hegehog ansatz gives us a Skyrmion solution we see that we have to find solutions of equations (18) with the boundary conditions:

$$f(-\infty, \rho) = 0, f(+\infty, \rho) = \pi; g(x, 0) = \pi, g(x, +\infty) = 0. \quad (19)$$

These boundary conditions guarantee a nontrivial topological charge of the system. Note that there are no nontrivial solutions of (18) within the class of functions with separated variables, *i.e.* of the form

$$f = f(x), g = g(\rho). \quad (20)$$

This can be easily verified by looking at the Lagrangian (16), in which the term

$$\sim (f_x g_\rho - f_\rho g_x)^2$$

clearly indicates that it is not possible to find any nontrivial solutions of the form (20).

Let us look next for the stationary solutions of eqs. (18). Consider the expression for the mass term  $V(f, g)$  of the form

$$V(f, g) = \frac{1}{2} \sin^2 f (1 - \delta_m \cos^2 g), \quad (21)$$

where  $0 < \delta < 1$ . Using this form of  $V(f, g)$ , and performing a numerical simulation, we have managed to find a solution of equations (18) with the boundary conditions (19).

This solution looks like a slightly deformed 2-dimensional (in  $\rho, \varphi$  variables) Skyrmion (or hegehog) located on the domain wall near the origin  $x = 0$ . The picture of the fields  $f$  and  $g$  as well as the energy density for this solution is shown in Figure 1 using the  $r, x$  coordinates. One sees clearly that it corresponds to a bound states of a Skyrmion and a domain wall. The topological charge of the configuration is 1.

The solution is stable with respect to the 3-dimensional ( $\vec{r} \rightarrow \epsilon \vec{r}$ ) scaling transformations as well as with respect to 2-dimensional ones,  $(x, \rho, \varphi) \rightarrow (x, \lambda \rho, \varphi)$ . Notice that the stability condition with respect to these 2-dimensional transformations is similar to the condition for Skyrmons in the baby-Skyrmion model in (2+1) dimensions, where a mass term is needed to guarantee their stability (see, e.g. [11] and references therein).

In the case when  $\delta_m \rightarrow 0$ , *i.e.* for the case when

$$V(f, g) = \frac{1}{2} \sin^2 f \quad (22)$$

any initially formed field configuration which carries a topological charge located on the domain wall is unstable with respect to the scale transformations in the  $(\rho, \varphi)$ -plane. This is why any initially formed field configuration in the case  $\delta_m = 0$  looks like a wave packet, which spreads along the domain wall. This shows that in the case of the Lagrangian (1) with the mass term given by (5) nontrivial topological configurations on the domain wall manifest themselves as topological radial waves. Thus the Skyrmion, located outside the domain wall, when captured, is absorbed by the domain wall. A necessary condition for this capture process is an attraction between the Skyrmion and the domain wall.

#### IV. ATTRACTION BETWEEN SKYRMIONS AND DOMAIN WALLS

In this section we show that Skyrmons can be absorbed by a domain wall and form field configurations described in the previous section. To demonstrate this we have performed numerical simulations of the time evolution of various field configurations describing Skyrmons and domain walls. In our work we used the Lagrangian (1) written in terms of three angular fields  $f, g$  and  $h$ . We were forced to consider this, a more general configuration than (13), as for an isolated domain wall the field  $h$  is constant but this is incompatible with the hegehog ansatz which describes the Skyrmion. Thus, to study their interaction we need



to consider the most general fields. We see that the problem of determining the interaction of Skyrmions and domain walls is more complicated than the problem of looking for bound states of Skyrmions and domain walls.

First we have performed the numerical evolution of a Skyrmion placed at a finite distance from the domain wall in the first modified model ( $\delta = 0$ ). We have done this for various relative orientations of the Skyrmion with respect to the domain wall. In each case the evolution was similar for every orientation that we tried: the Skyrmion was attracted by the wall to finally merge with it. The structure formed by this process was then dissipated by radiating concentric waves on the wall. Figure 2 shows the evolution of a cross section in the  $x - y$  plane (the plane perpendicular to the wall) of the energy density for this process. One sees that, after being absorbed, the Skyrmion is transformed into a wave which happens to be topologically non trivial.

Next we have performed the same simulation when  $\delta = 0.3$ . Again the Skyrmion was absorbed by the wall except that this time instead of being radiated along the wall, it formed a bound state with the wall. In Figure 3 we show the evolution of a cross section in the  $x - y$  plane of the energy density for this process. This configuration carries one unit of topological charge and is very similar to the topological structure described in the previous section.

We then looked at the interaction between a 2-Skyrmion configuration and the wall using  $\delta = 0.3$  again. As for the single Skyrmion, the two Skyrmions were attracted by the wall and merged with it to form a static ring-like structure on the wall. In Figure 4, we show the cross section of the energy density in the  $x - y$  plane and in the  $y - z$  plane (on the wall) after the structure had settled down.

## V. SUMMARY AND FINAL COMMENTS

We have established the existence of bound states of Skyrmions and domain walls. Depending on the form of the mass term in the theory this solution is either stable or unstable. In the case of  $O(4)$ -symmetry broken down to  $O(3) \times Z_2$  this solution is unstable and its evolution looks like an evolution of a wave packet on the domain wall which carries the topological charge. We have also shown that Skyrmions can be absorbed by the domain wall and that their field configuration is transmuted into topological waves on the wall. Considering

the topological charge as the baryonic charge, this picture corresponds to the absorption of baryons by the domain walls. Note that, because of the different vacua to the left and to the right of the domain wall, Skyrmions and antiskyrmions (or baryons and antibaryons) behave differently. We end with the speculation that these arguments, *ie* the separation of Skyrmions and antiskyrmions by the domain wall and the observation of the absorption of Skyrmions by the walls may be relevant to the discussion of the observed baryon-antibaryon asymmetry of the Universe.

### **Acknowledgments.**

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# FIGURES

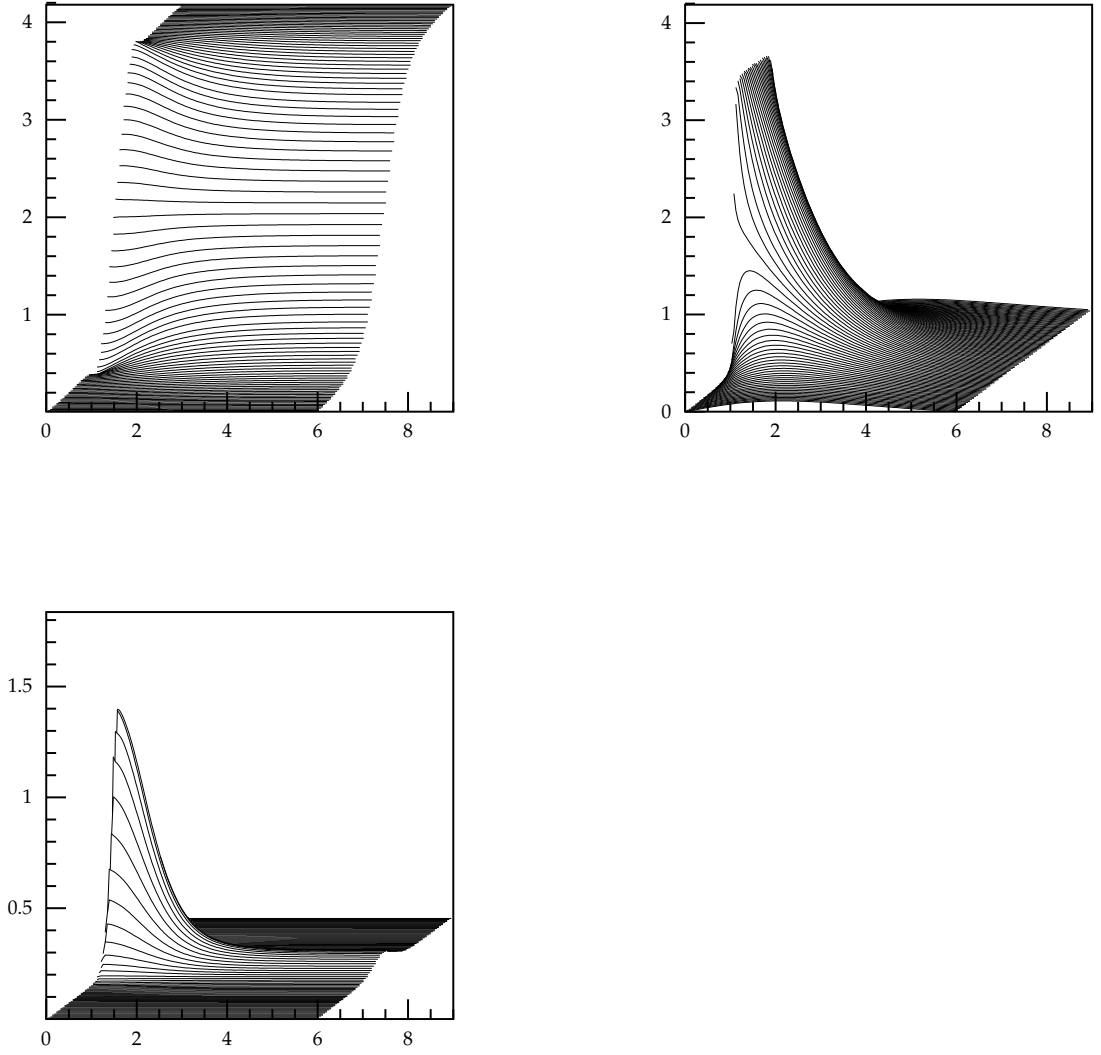


FIG. 1. Skyrmion on the Wall ( $\delta = 0.3$ ). All expressions are shown using the  $r, x$  coordinates:  
a)  $f$ ; b)  $g$ ; c) Energy density

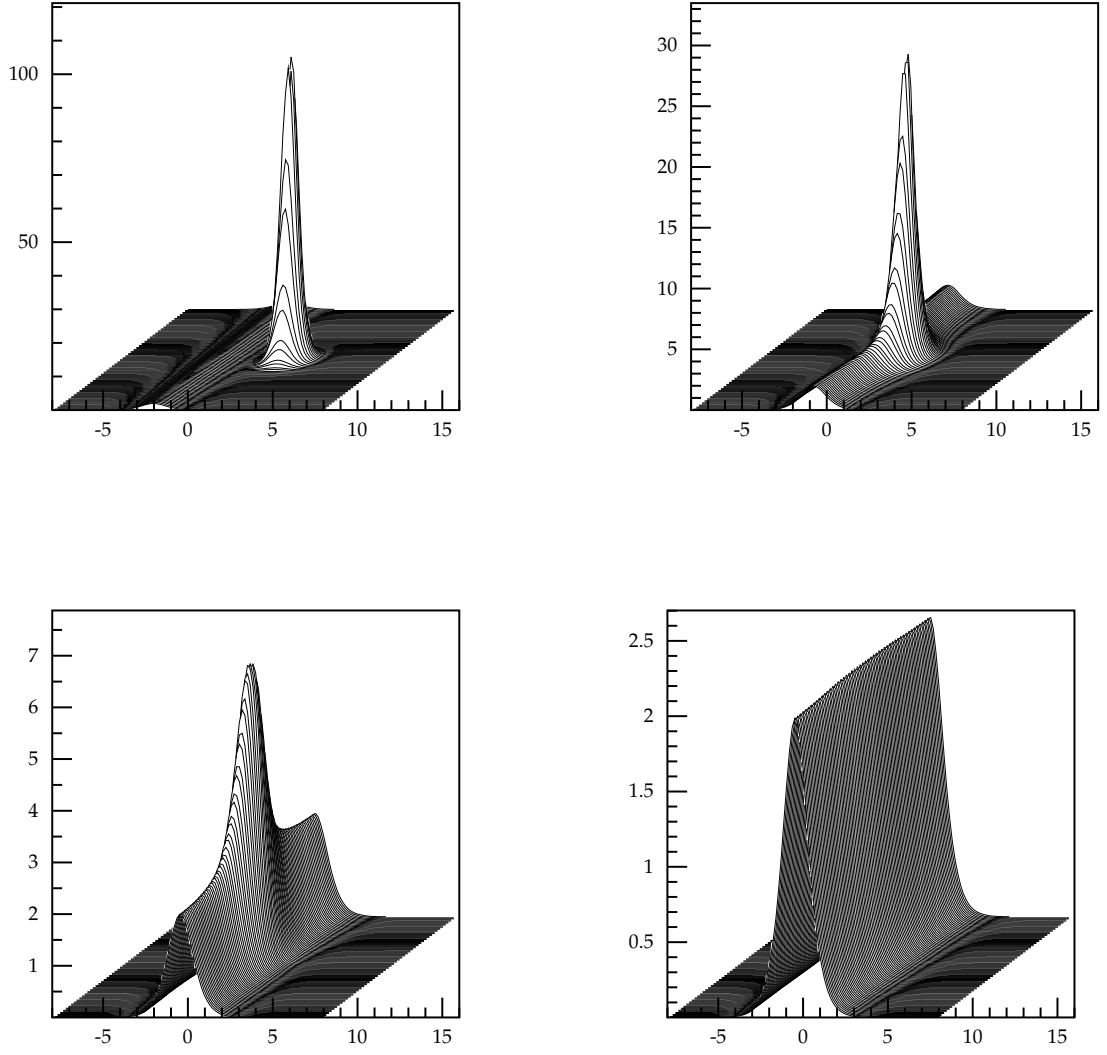


FIG. 2. Skyrmion absorption by the Wall ( $\delta = 0$ ). Energy density cross section perpendicular to the wall at: a)  $t=0$ ; b)  $t=37.5$ ; c)  $t=50$ ; d)  $t=65$

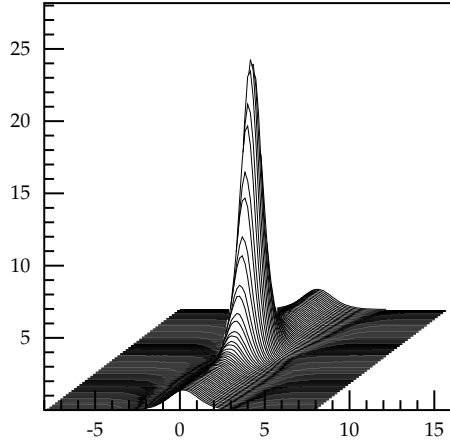


FIG. 3. Static Skymions on the Wall ( $\delta = 0.3$ ) : cross section of the energy density perpendicular to the wall

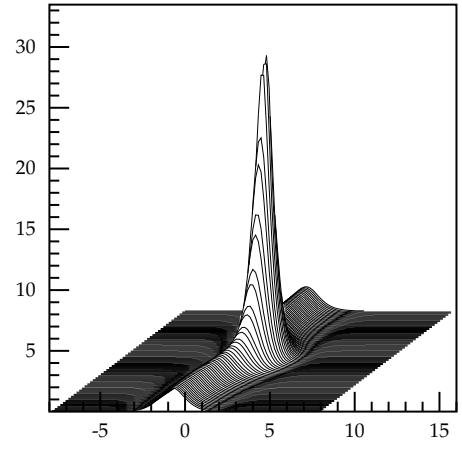
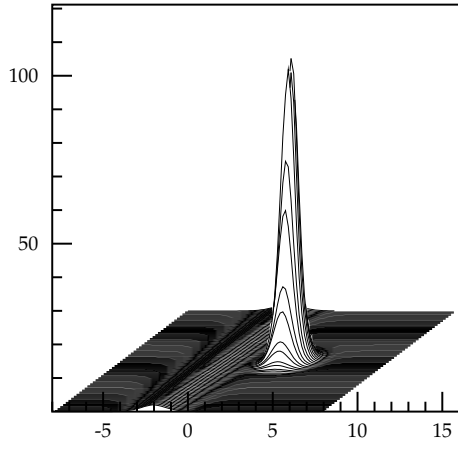


FIG. 4. Two Skymions on the Wall ( $\delta = 0.3$ ) : cross section of the energy density a) perpendicular to the wall b) parallel to the wall